

A NOTE ON LOCAL HÖLDER CONTINUITY OF WEIGHTED TAUBERIAN FUNCTIONS

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ABSTRACT. Let M and M_S respectively denote the Hardy-Littlewood maximal operator with respect to cubes and the strong maximal operator on \mathbb{R}^n , and let w be a nonnegative locally integrable function on \mathbb{R}^n . We define the associated Tauberian functions $C_{HL,w}(\alpha)$ and $C_{S,w}(\alpha)$ on $(0, 1)$ by

$$C_{HL,w}(\alpha) := \sup_{\substack{E \subset \mathbb{R}^n \\ 0 < w(E) < \infty}} \frac{1}{w(E)} w(\{x \in \mathbb{R}^n : M\chi_E(x) > \alpha\})$$

and

$$C_{S,w}(\alpha) := \sup_{\substack{E \subset \mathbb{R}^n \\ 0 < w(E) < \infty}} \frac{1}{w(E)} w(\{x \in \mathbb{R}^n : M_S\chi_E(x) > \alpha\}).$$

Utilizing weighted Solyanik estimates for M and M_S , we show that the function $C_{HL,w}$ lies in the local Hölder class $C^{(c_n[w]_{A_\infty})^{-1}}(0, 1)$ and $C_{S,w}$ lies in the local Hölder class $C^{(c_n[w]_{A_\infty^*})^{-1}}(0, 1)$, where the constant $c_n > 1$ depends only on the dimension n .

1. INTRODUCTION

This note concerns how Solyanik estimates may be used to establish local Hölder continuity estimates for the Tauberian functions associated to the Hardy-Littlewood and strong maximal operators in the context of Muckenhoupt weights. In [5], Hagelstein and Parissis used Solyanik estimates to prove that the Tauberian functions $C_{HL}(\alpha)$ and $C_S(\alpha)$ associated to the Hardy-Littlewood and strong maximal operators in \mathbb{R}^n both lie in the local Hölder class $C^{1/n}(1, \infty)$. The techniques of that paper are surprisingly robust, and we here will show how the weighted Solyanik estimates for the Hardy-Littlewood and strong maximal operators obtained in [6, 7] may be used to establish local Hölder smoothness estimates for the Tauberian functions of the Hardy-Littlewood and strong maximal operators in the weighted scenario.

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We now briefly review what Solyanik estimates are and how they may be used to establish local smoothness estimates for Tauberian functions associated to geometric maximal operators in the setting of Lebesgue measure. Let \mathcal{B} be a collection of sets of positive measure in \mathbb{R}^n , and define the associated geometric maximal operator $\mathbf{M}_{\mathcal{B}}$ by

$$\mathbf{M}_{\mathcal{B}}f(x) := \sup_{x \in R \in \mathcal{B}} \frac{1}{|R|} \int_R |f|.$$

For $0 < \alpha < 1$, the associated Tauberian function $\mathbf{C}_{\mathcal{B}}(\alpha)$ is given by

$$\mathbf{C}_{\mathcal{B}}(\alpha) := \sup_{\substack{E \subset \mathbb{R}^n \\ 0 < |E| < \infty}} \frac{1}{|E|} |\{x \in \mathbb{R}^n : \mathbf{M}_{\mathcal{B}}\chi_E(x) > \alpha\}|.$$

Our ordinary expectation is that, provided \mathcal{B} is a basis with reasonable differentiation properties, for $0 < \alpha < 1$ and α very close to 1, we should have $|\{x \in \mathbb{R}^n : \mathbf{M}_{\mathcal{B}}\chi_E(x) > \alpha\}|$ is very close to $|E|$ itself, and accordingly that $\mathbf{C}_{\mathcal{B}}(\alpha)$ is very close to 1. Solyanik estimates provide a quantitative validation of this expectation. In particular, we have the following theorem due to Solyanik [9]; see also [4].

Theorem 1 (Solyanik, [9]). *We have the following Solyanik estimates for the Hardy-Littlewood and the strong maximal operator:*

- (a) *Let \mathbf{M} denote the uncentered Hardy-Littlewood maximal operator on \mathbb{R}^n with respect to cubes, and define the associated Tauberian function $\mathbf{C}_{\text{HL}}(\alpha)$ by*

$$\mathbf{C}_{\text{HL}}(\alpha) = \sup_{\substack{E \subset \mathbb{R}^n \\ 0 < |E| < \infty}} \frac{1}{|E|} |\{x \in \mathbb{R}^n : \mathbf{M}\chi_E(x) > \alpha\}|.$$

Then for $\alpha \in (0, 1)$ sufficiently close to 1 we have

$$\mathbf{C}_{\text{HL}}(\alpha) - 1 \lesssim_n (1 - \alpha)^{1/n}.$$

- (b) *Let \mathbf{M}_S denote the strong maximal operator on \mathbb{R}^n , and define the associated Tauberian function $\mathbf{C}_S(\alpha)$ by*

$$\mathbf{C}_S(\alpha) := \sup_{\substack{E \subset \mathbb{R}^n \\ 0 < |E| < \infty}} \frac{1}{|E|} |\{x \in \mathbb{R}^n : \mathbf{M}_S\chi_E(x) > \alpha\}|.$$

Then for $\alpha \in (0, 1)$ sufficiently close to 1 we have

$$\mathbf{C}_S(\alpha) - 1 \lesssim_n (1 - \alpha)^{1/n}.$$

The following theorem associated to the embedding of so-called halo sets enables us to relate Solyanik estimates to Hölder smoothness estimates.

Theorem 2 (Hagelstein, Parissis, [5]). *Let \mathcal{B} be a homothecy invariant collection of rectangular parallelepipeds in \mathbb{R}^n . Given a set $E \subset \mathbb{R}^n$ of finite measure and $0 < \alpha < 1$, define the associated halo set $\mathcal{H}_\alpha(E)$ by*

$$\mathcal{H}_{\mathcal{B},\alpha}(E) := \{x \in \mathbb{R}^n : \mathbf{M}_{\mathcal{B}}\chi_E(x) > \alpha\}.$$

Then for all $\alpha, \delta \in (0, 1)$ with $\alpha < 1 - \delta$, we have

$$\mathcal{H}_{\mathcal{B},\alpha}(E) \subset \mathcal{H}_{\mathcal{B},\alpha(1+2^{-(n+1)}\delta)}(\mathcal{H}_{\mathcal{B},1-\delta}(E)).$$

An immediate corollary of this theorem is the following.

Corollary 1 (Hagelstein, Parissis, [5]). *Let \mathcal{B} be a homothecy invariant collection of rectangular parallelepipeds in \mathbb{R}^n and let $\alpha, \delta \in (0, 1)$. Then for $\alpha < 1 - \delta$ we have*

$$C_{\mathcal{B}}(\alpha) \leq C_{\mathcal{B}}(\alpha(1 + 2^{-(n+1)}\delta))C_{\mathcal{B}}(1 - \delta).$$

Now, we of course have that $C_{\mathcal{B}}(\alpha)$ is nonincreasing on $(0, 1)$. If \mathcal{B} is the collection of rectangular parallelepipeds in \mathbb{R}^n whose sides are parallel to the axes (so that $\mathbf{M}_{\mathcal{B}} = \mathbf{M}_{\mathcal{S}}$), we can accordingly combine the above corollary with the Solyanik estimates for $\mathbf{M}_{\mathcal{S}}$ provided by Theorem 1 to relatively easily obtain the following.

Corollary 2 (Hagelstein, Parissis, [5]). *Let $C_{\text{HL}}(\alpha)$ and $C_{\mathcal{S}}(\alpha)$ respectively denote the Tauberian functions associated to the Hardy-Littlewood maximal operator with respect to cubes and the strong maximal operator in \mathbb{R}^n with respect to α . Then*

$$C_{\text{HL}} \in C^{1/n}(0, 1) \quad \text{and} \quad C_{\mathcal{S}} \in C^{1/n}(0, 1).$$

The purpose of this note is to establish weighted analogues of Corollary 2. To make this precise let us consider a non-negative, locally integrable function w on \mathbb{R}^n . The relevant Tauberian functions $C_{\text{HL},w}(\alpha)$ and $C_{\mathcal{S},w}(\alpha)$ are defined on $(0, 1)$ by

$$C_{\text{HL},w}(\alpha) := \sup_{\substack{E \subset \mathbb{R}^n \\ 0 < w(E) < \infty}} \frac{1}{w(E)} w(\{x \in \mathbb{R}^n : \mathbf{M}\chi_E(x) > \alpha\})$$

and

$$C_{\mathcal{S},w}(\alpha) := \sup_{\substack{E \subset \mathbb{R}^n \\ 0 < w(E) < \infty}} \frac{1}{w(E)} w(\{x \in \mathbb{R}^n : \mathbf{M}_{\mathcal{S}}\chi_E(x) > \alpha\}).$$

It was shown in [3] that the condition $C_{\text{HL},w}(\alpha) < +\infty$ for *some* $\alpha \in (0, 1)$ already implies that $\mathbf{M} : L^p(w) \rightarrow L^p(w)$ for *some* $1 < p < \infty$ and, similarly if $C_{\mathcal{S},w}(\alpha) < +\infty$ for *some* $\alpha \in (0, 1)$ then $\mathbf{M}_{\mathcal{S}} : L^p(w) \rightarrow L^p(w)$ for *some* $1 < p < \infty$. These results pose an important restriction on the kind of functions w we can consider in proving Hölder regularity estimates for $C_{\text{HL},w}$ and $C_{\mathcal{S},w}$. In particular, it is well known that the class of functions w such that $\mathbf{M} : L^p(w) \rightarrow L^p(w)$ for some $p \in (1, \infty)$ is the Muchkenhoupt

class of weights A_∞ ; see for example [2]. Here we use the Fujii-Wilson definition of the Muckenhoupt class A_∞ . Namely, the weight w belongs to the class A_∞ if and only if

$$[w]_{A_\infty} := \sup_Q \frac{1}{w(Q)} \int_Q M(w\chi_Q) < +\infty,$$

where the supremum is taken with respect to all cubes in \mathbb{R}^n whose sides are parallel to the axes. This description of the class A_∞ goes back to Fujii, [8], and Wilson, [10, 11]; see also [1]. Thus $w \in A_\infty$ is a necessary condition for the continuity of $C_{HL,w}$ on $(0, 1)$. It turns out that $w \in A_\infty$ is also a sufficient condition for the Hölder regularity of $C_{HL,w}$.

Theorem 3. *Let $w \in A_\infty$ be a Muckenhoupt weight on \mathbb{R}^n . Then*

$$C_{HL,w} \in C^{(c_n[w]_{A_\infty})^{-1}}(0, 1),$$

where the constant c_n depends only on the dimension n .

Moving to the multiparameter case, the condition that $M_S : L^p(w) \rightarrow L^p(w)$ for some $p \in (1, \infty)$ is equivalent to the condition $w \in A_\infty^*$, where A_∞^* denotes the class of *multiparameter* or *strong* Muckenhoupt weights. A few words about how the multiparameter Muckenhoupt class A_∞^* is defined are in order here. For $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ and $1 \leq j \leq n$ we may associate the point $\bar{x}^j := (x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_n) \in \mathbb{R}^{n-1}$. Associated to a non-negative locally integrable function w on \mathbb{R}^n and \bar{x}^j is the one-dimensional weight

$$w_{\bar{x}^j}(t) := w(x_1, \dots, x_{j-1}, t, x_{j+1}, \dots, x_n), \quad t \in \mathbb{R}.$$

Then $[w]_{A_\infty^*}$ is defined by

$$[w]_{A_\infty^*} := \sup_{1 \leq j \leq n} \operatorname{ess\,sup}_{\bar{x}^j \in \mathbb{R}^{n-1}} [w_{\bar{x}^j}]_{A_\infty}.$$

Here $[\nu]_{A_\infty}$ denotes the standard Fujii-Wilson A_∞ constant of a weight ν on \mathbb{R}^1 , given by

$$[\nu]_{A_\infty} := \sup_I \frac{1}{w(I)} \int_I M_1(\nu\chi_I),$$

where the supremum is taken over all intervals $I \subset \mathbb{R}$ and M_1 denotes the Hardy-Littlewood maximal operator on \mathbb{R}^1 . Thus a weight w is a multiparameter Muckenhoupt weight if and only if $[w]_{A_\infty^*} < +\infty$. We refer the reader to [7] and the references therein for more details on the definition and properties of multiparameter Muckenhoupt weights.

With the definition of multiparameter Muckenhoupt weights in hand, the previous discussion shows that a necessary condition for the continuity of $C_{S,w}$ on $(0, 1)$ is that $w \in A_\infty^*$. As in the one parameter case, we show that $w \in A_\infty^*$ is also sufficient for the Hölder continuity of $C_{S,w}$ on $(0, 1)$.

Theorem 4. *Let $w \in A_\infty^*$ be a multiparameter Muckenhoupt weight on \mathbb{R}^n . Then*

$$C_{S,w} \in C^{(c_n[w]_{A_\infty^*})^{-1}}(0, 1),$$

where the constant c_n depends only on the dimension n .

2. NOTATION

We use the letters C, c to denote positive numerical constants whose value might change even in the same line of text. We express the dependence of a constant C on some parameter n by writing C_n . We write $A \lesssim B$ if $A \leq CB$ for some numerical constant $C > 0$. If $A \leq C_n B$ we then write $A \lesssim_n B$. In this note, w will always denote a non-negative, locally integrable function on \mathbb{R}^n . Finally, we say that a function f lies in the Hölder class $C^p(I)$ for some interval $I \subset \mathbb{R}$ if for every compact set $K \subset I$ we have $|f(x) - f(y)| \lesssim_K |x - y|^p$ for all $x, y \in K$. In this case we will say that f is *locally Hölder continuous* with exponent p in I .

3. WEIGHED SOLYANIK ESTIMATES AND HÖLDER REGULARITY

In this section we show that the strategy for establishing Hölder smoothness estimates for $C_{HL}(\alpha)$ and $C_S(\alpha)$ may be adapted to the weighted context. To implement the above strategy, we need Solyanik estimates that provide us quantitative information as to how close $C_{HL,w}(\alpha)$ and $C_S(\alpha)$ are to 1 for α near 1. Of course, the related estimates are expected to depend on w . Suitable Solyanik estimates in this regard were found in [6, 7] when w is a Muckenhoupt weight. In particular, we have the following:

Theorem 5 (Hagelstein, Parissis, [6, 7]). *Let $w \in A_\infty$. We have the Solyanik estimate*

$$C_{HL,w}(\alpha) - 1 \lesssim_n \Delta_w^2 (1 - \alpha)^{(c_n[w]_{A_\infty})^{-1}} \quad \text{whenever } 1 > \alpha > 1 - e^{-c_n[w]_{A_\infty}}.$$

Here Δ_w is the doubling constant of w , and c_n and the implied constant depend only upon the dimension n .

A multiparameter analogue of Theorem 5 the following.

Theorem 6 (Hagelstein, Parissis [7]). *Let w be a non-negative, locally integrable function in \mathbb{R}^n . If $w \in A_\infty^*$ we have*

$$C_{S,w}(\alpha) - 1 \lesssim_n (1 - \alpha)^{(c_n[w]_{A_\infty^*})^{-1}} \quad \text{for all } 1 > \alpha > 1 - e^{-c_n[w]_{A_\infty^*}},$$

where $c > 0$ is a numerical constant.

With these *weighted Solyanik estimates* at our disposal we can now give the proof of the Hölder continuity estimates for $C_{HL,w}$ and $C_{S,w}$.

Proof of Theorem 3. Let K be a compact subset in $(0, 1)$ and let $m_K, M_K \in (0, 1)$ be such that $m_K \leq x \leq M_K$ for all $x \in K$. Since $w \in A_\infty$ there exists some $q \in (0, 1)$ such that $\mathbf{M} : L^q(w) \rightarrow L^{q,\infty}(w)$ and thus $\sup_{\alpha \in K} \mathbf{C}_{\text{HL},w}(\alpha) \lesssim_{w,n,K} 1$. Furthermore, by Theorem 5 we have that

$$(1) \quad \mathbf{C}_{\text{HL},w}(\alpha) - 1 \lesssim_{w,n} (1 - \alpha)^{(c_n[w]_{A_\infty})^{-1}} \quad \text{for all } 1 > \alpha > 1 - e^{-c_n[w]_{A_\infty}} =: \alpha_o.$$

We first consider $x, y \in K$ with $0 < y - x < \min(\frac{1-M_K}{2^{n+1}}m_K, \frac{1-\alpha_o}{2^{n+1}}m_K) =: \eta$. We can then write

$$\mathbf{C}_{\text{HL},w}(x) - \mathbf{C}_{\text{HL},w}(y) = \mathbf{C}_{\text{HL},w}(x) - \mathbf{C}_{\text{HL},w}\left(x\left(1 + 2^{n+1}\frac{y-x}{2^{n+1}x}\right)\right).$$

Now observe that by our choice of x, y we have

$$2^{n+1}\frac{y-x}{x} < 2^{n+1}\frac{1-M_K}{2^{n+1}}m_K\frac{1}{m_K} \leq 1 - M_K \leq 1 - x.$$

We can thus apply Theorem 2 with x in the role of $\alpha := x$ and $\delta := 2^{n+1}\frac{y-x}{x}$ to get

$$\mathcal{H}_{\mathcal{B},x}(E) \subset \mathcal{H}_{\mathcal{B},y}(\mathcal{H}_{\mathcal{B},(1-\delta)}(E))$$

for all measurable E where here \mathcal{B} denotes the collection of all cubes in \mathbb{R}^n whose sides are parallel to the axes. This immediately implies

$$\mathbf{C}_{\text{HL},w}(x) \leq \mathbf{C}_{\text{HL},w}(y)\mathbf{C}_{\text{HL},w}\left(1 - 2^{n+1}\frac{y-x}{x}\right).$$

Thus we can estimate

$$\begin{aligned} \mathbf{C}_{\text{HL},w}(x) - \mathbf{C}_{\text{HL},w}(y) &\leq \mathbf{C}_{\text{HL},w}(y) \left[\mathbf{C}_{\text{HL},w}\left(1 - 2^{n+1}\frac{y-x}{x}\right) - 1 \right] \\ &\lesssim_{w,n,K} \mathbf{C}_{\text{HL},w}\left(1 - 2^{n+1}\frac{y-x}{x}\right) - 1 \end{aligned}$$

since $\sup_{\alpha \in K} \mathbf{C}_{\text{HL},w}(\alpha) \lesssim_{w,n,K} 1$. Noting that

$$1 > 1 - 2^{n+1}\frac{y-x}{x} > 1 - 2^{n+1}\frac{1-\alpha_o}{2^{n+1}x}m_K \geq \alpha_o,$$

an appeal to (1) gives

$$\mathbf{C}_{\text{HL},w}(x) - \mathbf{C}_{\text{HL},w}(y) \lesssim_{w,n,K} \left(\frac{y-x}{x}\right)^{(c_n[w]_{A_\infty})^{-1}} \lesssim_K (y-x)^{(c_n[w]_{A_\infty})^{-1}}.$$

We have shown that

$$\sup_{\substack{x,y \in K \\ |y-x| < \eta}} \frac{|\mathbf{C}_{\text{HL},w}(y) - \mathbf{C}_{\text{HL},w}(x)|}{|y-x|^{(c_n[w]_{A_\infty})^{-1}}} \lesssim_{w,n,K} 1.$$

On the other hand, if $x, y \in K$ with $y - x \geq \eta$ then the Hölder estimate follows trivially since $\sup_{x,y \in K} |\mathbf{C}_{\text{HL},w}(x) - \mathbf{C}_{\text{HL},w}(y)| \lesssim_{w,n,K} 1$ so we are done. \square

The proof of Theorem 4 is virtually identical to that of Theorem 3.

One may naturally wonder how sharp the above smoothness estimates are for $C_{HL,w}(\alpha)$ and $C_{S,w}(\alpha)$. In particular we may ask the questions: Are $C_{HL,w}(\alpha)$ and $C_{S,w}(\alpha)$ differentiable on $(0, 1)$? Are they in fact smooth on $(0, 1)$? To the best of our knowledge, even the question of whether or not the sharp Tauberian constant $C_{HL}(\alpha)$ of the Hardy-Littlewood maximal operator on \mathbb{R} in the Lebesgue setting is differentiable constitutes an unsolved problem. All of these topics remain a subject of continuing research.

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